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ITERATIVE MAXIMUM-LIKELIHOOD ESTIMATION OF THE PARAMETERS OF NORMAL POPULATIONS FROM SINGLY AND DOUBLY CENSORED SAMPLES

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Iterative maximum-likelihood estimation of the parameters of normal populations from singly and doubly censored samples

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SUMMARY

Iterative procedures are given for joint maximum-likelihood estimation based on singly and doubly censored samples from a normal population. The simultaneous equations yielding the maximum-likelihood estimates are obtained. Since their algebraic solution is impossible, iterative procedures are proposed which are applicable in the most general case in which both parameters are unknown and in special cases in which either of the parameters is known. The asymptotic variances and covariances are tabulated for 10% censoring intervals. A Monte Carlo investigation of the means and standard deviations of the maximum-likelihood estimators was made for 1000 samples from the standard normal population for $n = 10$ and $n = 20$. A comparison was then made of best linear unbiased estimators and maximum-likelihood estimators for $n = 10$ and $n = 20$.

1. INTRODUCTION

The estimation of the parameters of a censored sample from a normal population has been considered by many authors, who have used several different methods including the method of least squares and the method of maximum likelihood.

Lloyd (1952) applied the theory of least-squares estimation to an ordered sample from distributions depending on location and scale parameters only. Gupta (1952) derived best linear estimators ($n \leqslant 10$) for the mean and variance using singly censored samples from normal populations and for larger values of n derived an alternative linear estimator. Sarhan & Greenberg (1956, 1958*a, b*) estimated the mean and standard deviation of normal populations from singly and doubly censored samples ($n \leqslant 20$) by the method of least squares. Saw (1959) developed simplified unbiased estimators of the mean and variance given a singly censored sample from a normal population ($n \leqslant 20$). Dixon (1957, 1960) derived simplified estimators of the mean and standard deviation for complete and censored normal samples which are almost as efficient as the best linear estimators ($n \leqslant 20$). Walsh (1956) obtained distribution-free estimators for the population mean and variance for a rather general class of continuous statistical populations using doubly censored samples.

Cohen (1950) used the method of maximum likelihood to estimate the parameters of normal populations from singly and doubly truncated samples. The term 'truncated samples' was used by Cohen in a sense somewhat broader than its present usage and included what are now called 'censored samples'. Cohen was primarily concerned, however, with Type I censoring (at a specified time) rather than Type II censoring (when a specified number of failures have occurred). Gupta (1952) found maximum-likelihood equations for estimators of the parameters of a normal population from a sample censored from above

(Type II censoring), and determined their asymptotic variances and covariances. Halperin (1952) proved under mild regularity conditions that the maximum-likelihood estimator of a single parameter from singly censored samples is consistent, asymptotically normally distributed, and of minimum variance for large samples and indicated his results could be generalized to several parameters and more general censoring. Breakwell (1953) also obtained maximum-likelihood estimators for singly censored samples, asymptotic distributions of the estimators, and their asymptotic biases. Plackett (1958) showed that maximum-likelihood estimators are asymptotically linear and that the best linear unbiased estimators are asymptotically normal and efficient. Plackett computed a 'linearized maximum-likelihood' estimator and compared it with the best linear unbiased estimator for the standard deviation of a normal population from censored samples ($n \leq 10$). In three later papers Cohen (1955, 1959, 1961) extended the results given in his 1950 paper. The present paper in part duplicates the work of Cohen and Gupta but extends the results for Type II censoring to include maximum-likelihood estimation of the parameters of a normal population from a doubly censored sample, together with a completely computerized iterative procedure, and mathematical expressions and tables for asymptotic variances and covariances. The mathematical formulation for maximum-likelihood estimation is given in § 2, the asymptotic variances and covariances of the estimators are given in § 3. A discussion of the iterative procedures for maximum-likelihood estimation is given in § 4. A Monte Carlo study of the maximum-likelihood estimators together with a comparison with the best linear estimator is given, for small samples, in § 5.

2. NORMAL POPULATION–MATHEMATICAL FORMULATION

Consider a random sample of size n from a normal population with mean μ and standard deviation σ and let $X_{r_1+1}, \ldots, X_{n-r_2}$ be the ordered observations remaining when the r_1 smallest observations and the r_2 largest observations have been censored. The joint probability density function of these order statistics is given by

$$f(x_{r_1+1}, \ldots, x_{n-r_2}; \mu, \sigma) = \frac{n!}{r_1!\, r_2!} \{\sqrt{(2\pi)}\,\sigma\}^{-m} \exp\left[\sum_{i=r_1+1}^{n-r_2} \{-(x_i-\mu)^2/(2\sigma^2)\}\right] \times \left[F\left(\frac{x_{r_1+1}-\mu}{\sigma}\right)\right]^{r_1} \left[1-F\left(\frac{x_{n-r_2}-\mu}{\sigma}\right)\right]^{r_2}, \tag{2·1}$$

where $m = n - r_1 + r_2$.

The natural logarithm of the likelihood function is given by

$$L = \ln \frac{n!}{r_1!\, r_2!} - \tfrac{1}{2} m \ln 2\pi - m \ln \sigma - \sum_{i=r_1+1}^{n-r_2} \frac{(x_i-\mu)^2}{2\sigma^2} + r_1 \ln F\left(\frac{x_{r_1+1}-\mu}{\sigma}\right) + r_2 \ln\left[1 - F\left(\frac{x_{n-r_2}-\mu}{\sigma}\right)\right]. \tag{2.2}$$

The likelihood equations are:

$$\frac{\partial L}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=r_1+1}^{n-r_2} (x_i-\mu) - \frac{r_1}{\sigma} \frac{f[(x_{r_1+1}-\mu)/\sigma]}{F[(x_{r_2+1}-\mu)/\sigma]} + \frac{r_2}{\sigma} \frac{f[(x_{n-r_2}-\mu)/\sigma]}{1-F[(x_{n-r_2}-\mu)/\sigma]} = 0, \tag{2·3}$$

$$\frac{\partial L}{\partial \sigma} = -\frac{m}{\sigma} + \frac{1}{\sigma^3} \sum_{i=r_1+1}^{n-r_2} (x_i-\mu)^2 - r_1 \frac{[(x_{r_1+1}-\mu)/\sigma^2]\, f[(x_{r_1+1}-\mu)/\sigma}{F[(x_{r_2+1}-\mu)/\sigma]} + r_2 \frac{[(x_{n-r_2}-\mu)/\sigma^2]\, f[(x_{n-r_2}-\mu)/\sigma]}{1-F[(x_{n-r_2}-\mu)/\sigma]} = 0. \tag{2·4}$$

If $m = n$, i.e. if $r_1 = r_2 = 0$, these equations have explicit solutions

$$\hat{\mu} = \sum_{i=1}^{n} x_i/n, \quad \hat{\sigma} = \sqrt{\left\{\sum_{i=1}^{n} (x_i - \mu)^2/n\right\}}.$$

The details of the iterative procedure for determining the maximum-likelihood estimates will be given in § 5.

3. Asymptotic variances and covariances of normal estimators

Gupta (1952) has given theoretical expressions and a table for the asymptotic variances and covariances of the maximum-likelihood estimators of the parameters of a normal population from singly censored (from above) samples. His results will be extended in this section to the case of doubly censored samples.

The natural logarithm of the likelihood function of a sample of size n, from a normal population with mean μ and standard deviation σ, the lowest r_1 and the highest r_2 sample values having been censored, is given by

$$L = \ln \frac{n!}{r_1!\, r_2!} - \tfrac{1}{2} m \ln 2\pi - m \ln \sigma - \sum_{i=r_1+1}^{n-r_2} \frac{(x_i - \mu)^2}{2\sigma^2} + r_1 \ln F(z_1) + r_2 \ln [1 - F(z_2)], \tag{3·1}$$

where $$z_1 = (x_{r_1+1} - \mu)/\sigma, \quad z_2 = (x_{n-r_2} - \mu)/\sigma.$$

$$F(z_i) = \int_{-\infty}^{z_i} f(t)\, dt, \quad f(z_i) = \frac{1}{\sqrt{(2\pi)}} \exp(-\tfrac{1}{2} z_i^2).$$

and $m = n - r_1 - r_2$. In this notation, the first partial derivatives of L are given by

$$\frac{\partial L}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=r_1+1}^{n-r_2} (x_i - \mu) - \frac{r_1}{\sigma} \frac{f(z_1)}{F(z_1)} + \frac{r_2}{\sigma} \frac{z_2 f(z_2)}{1 - F(z_2)}, \tag{3.2}$$

$$\frac{\partial L}{\partial \sigma} = -\frac{m}{\sigma} + \frac{1}{\sigma^3} \sum_{i=r_1+1}^{n-r_2} (x_i - \mu)^2 - \frac{r_1 z_1 f(z_1)}{\sigma\; F(z_1)} + \frac{r_2}{\sigma} \frac{z_2 f(z_2)}{1 - F(z_2)}. \tag{3·3}$$

The second partial derivatives of L are given by

$$\frac{\partial^2 L}{\partial \mu^2} = -\frac{m}{\sigma^2} - \frac{r_1}{\sigma^2} \frac{f(z_1)}{F(z_1)} \left[z_1 + \frac{f(z_1)}{F(z_1)}\right] + \frac{r_2}{\sigma^2} \frac{f(z_2)}{1 - F(z_2)} \left[z_2 - \frac{f(z_2)}{1 - F(z_2)}\right], \tag{3·4}$$

$$\frac{\partial^2 L}{\partial \mu\, \partial \sigma} = -\frac{2}{\sigma^3} \sum_{i=r_1+1}^{n-r_2} (x_i - \mu) - \frac{r_1}{\sigma^2} \frac{f(z_1)}{F(z_1)} \left[z_1^2 + z_1 \frac{f(z_1)}{F(z_1)} - 1\right] + \frac{r_2}{\sigma^2} \frac{f(z_2)}{1 - F(z_2)} \left[z_2^2 - z_2 \frac{f(z_2)}{1 - F(z_2)} - 1\right], \tag{3·5}$$

$$\frac{\partial^2 L}{\partial \sigma^2} = \frac{m}{\sigma^2} - \frac{3}{\sigma^4} \sum_{i=r_1+1}^{n-r_2} (x_i - \mu)^2 - \frac{r_1}{\sigma^2} \frac{z_1 f(z_1)}{1 - F(z_1)} \left[z_1^2 + z_1 \frac{f(z_1)}{F(z_1)} - 2\right] + \frac{r_2}{\sigma^2} \frac{z_2 f(z_2)}{1 - F(z_2)} \left[z_2^2 - z_2 \frac{f(z_2)}{1 - F(z_2)} - 2\right]. \tag{3·6}$$

Now let $q_1 = r_1/n$, $q_2 = r_2/n$, and $p = 1 - q_1 - q_2 = m/n$. As $n \to \infty$ (q_1 and q_2 fixed), $z_1 \to \hat{z}_1$ where $\int_{-\infty}^{\hat{z}_1} f(t)\, dt = q_1$, $z_2 \to \hat{z}_2$ where $\int_{\hat{z}_2}^{\infty} f(t)\, dt = q_2$,

$$E\left(\sum_{i=r_1+1}^{n-r_2} \frac{x_i - \mu}{\sigma}\right) \to n \int_{\hat{z}_1}^{\hat{z}_2} t f(t)\, dt = -n[f(\hat{z}_2) - f(\hat{z}_1)],$$

and $$E\left(\sum_{i=r_1+1}^{n-r_2} \left(\frac{x_i - \mu}{\sigma}\right)^2\right) \to n \int_{\hat{z}_1}^{\hat{z}_2} t^2 f(t)\, dt = n[p - \{\hat{z}_2 f(\hat{z}_2) - \hat{z}_1 f(\hat{z}_1)\}].$$

The elements of the information matrix (multiplied by σ^2/n) may be written as

$$\lim_{n\to\infty} -\frac{\sigma^2}{n} E\left[\frac{\partial^2 L}{\partial\mu^2}\right] = p + f(\xi_1)\left[\xi_1 + \frac{f(\xi_1)}{q_1}\right] - f(\xi_2)\left[\xi_2 - \frac{f(\xi_2)}{q_2}\right] = v_{11}, \tag{3·7}$$

$$\lim_{n\to\infty} -\frac{\sigma^2}{n} E\left[\frac{\partial^2 L}{\partial\mu\,\partial\sigma}\right] = f(\xi_1) - f(\xi_2) + \xi_1 f(\xi_1)\left[\xi_1 + \frac{f(\xi_1)}{q_1}\right] - \xi_2 f(\xi_2)\left[\xi_2 - \frac{f(\xi_2)}{q_2}\right] = v_{12}, \tag{3·8}$$

$$\lim_{n\to\infty} -\frac{\sigma^2}{n} E\left[\frac{\partial^2 L}{\partial\sigma^2}\right] = 2p + \xi_1 f(\xi_1) - \xi_2 f(\xi_2) + \xi_1^2 f(\xi_1)\left[\xi_1 + \frac{f(\xi_1)}{q_1}\right] - \xi_2 f(\xi_2)\left[\xi_2^2 - \frac{f(\xi_2)}{q_2}\right] = v_{22}. \tag{3·9}$$

The asymptotic variance-covariance matrix for the estimators $\hat{\mu}$ and $\hat{\sigma}$ is then $\sigma^2[\sigma_{ij}]/n$, where $[\sigma_{ij}] = [v_{ij}]^{-1}$. If one drops the terms involving ξ_2 from the equations (3·7)–(3·9) the results agree with those given by Gupta for the case of single censoring.

Table 1. *Coefficients of σ^2/n in asymptotic variances and covariances of maximum-likelihood estimators of parameters μ and σ of normal population from samples of size n with proportions q_1 censored from below and q_2 from above*

		Both parameters unknown			σ known	μ known
q_1	q_2	$n\,\mathrm{var}(\hat{\mu})/\sigma^2$	$n\,\mathrm{cov}(\hat{\mu}, \hat{\sigma})/\sigma^2$	$n\,\mathrm{var}(\hat{\sigma})/\sigma^2$	$n\,\mathrm{var}(\hat{\mu})/\sigma^2$	$n\,\mathrm{var}(\hat{\sigma})/\sigma^2$
0·0	0·0	1·000000	0·000000	0·500000	1·000000	0·500000
0·0	0·1	1·020092	0·041136	0·585925	1·017205	0·584266
0·0	0·2	1·062323	0·106905	0·688692	1·045728	0·677934
0·0	0·3	1·138257	0·206568	0·819749	1·086204	0·782262
0·0	0·4	1·272656	0·359824	0·994759	1·142501	0·893025
0·0	0·5	1·517094	0·605233	1·241453	1·222031	1·000000
0·0	0·6	1·990850	1·025933	1·615494	1·339322	1·086805
0·0	0·7	3·019940	1·832190	2·247997	1·520647	1·136413
0·0	0·8	5·780392	3·717327	3·537484	1·874080	1·146899
0·0	0·9	17·794599	10·620022	7·513928	2·784491	1·175776
0·1	0·1	1·035011	0·000000	0·702692	1·035011	0·702692
0·1	0·2	1·070615	0·071658	0·847527	1·064557	0·842731
0·1	0·3	1·140391	0·187749	1·041120	1·106533	1·010210
0·1	0·4	1·274494	0·370562	1·315918	1·165014	1·202879
0·1	0·5	1·542208	0·715075	1·736943	1·247822	1·405385
0·1	0·6	2·128202	1·364988	2·458665	1·370365	1·583087
0·1	0·7	3·665653	2·880735	3·954475	1·567111	1·690586
0·1	0·8	9·774446	8·237227	8·655663	1·935427	1·713898
0·2	0·2	1·095839	0·000000	1·052478	1·095839	1·052478
0·2	0·3	1·152548	0·127812	1·341466	1·140370	1·327293
0·2	0·4	1·275501	0·360575	1·783003	1·202582	1·681071
0·2	0·5	1·556437	0·820702	2·537708	1·291020	2·104956
0·2	0·6	2·301737	1·897104	4·093984	1·422641	2·530381
0·2	0·7	5·184839	5·628780	8·927375	1·635853	2·816625
0·3	0·3	1·188673	0·000000	1·796338	1·188673	1·796338
0·3	0·4	1·285467	0·273191	2·569779	1·256424	2·511720
0·3	0·5	1·565414	0·938941	4·165856	1·353277	3·592676
0·3	0·6	2·689726	3·281978	9·043125	1·498614	5·038488
0·4	0·4	1·332365	0·000000	4·173987	1·332365	4·173987
0·4	0·5	1·569895	1·079093	9·089706	1·441790	8·347974

Interchanging q_1 and q_2 leaves variances and absolute value of covariance unchanged, but changes sign of covariance.

The computation of the elements c_{ij} of the information matrix (multiplied by σ^2/n), as given by equations (3·7)–(3·9), and the inversion of this matrix to obtain the coefficients of σ^2/n in the variance covariance matrix were performed on the IBM 1620 computer. The resulting coefficients of σ^2/n in $\text{var}(\hat{\mu})$, $\text{cov}(\hat{\mu}, \hat{\sigma})$, $\text{var}(\hat{\sigma})$, $\text{var}(\hat{\mu}|\sigma)$ and $\text{var}(\hat{\sigma}|\mu)$ are given in Table 1 for all combinations of q_1 and q_2 which are integral multiples of 0·1 and which are such that $q_1 + q_2 < 1$, and $q_1 \leqslant q_2$. Only half of the table is given since interchanging the values for q_1 and q_2 would produce no change in the tabular values except that $\text{cov}(\hat{\mu}, \hat{\sigma})$ changes sign. Values are given to six decimal places. The results for single censoring from above (first ten lines of Table 1), when rounded to five decimal places, agree with those of Gupta, except for slight discrepancies in the case $q_1 = 0·0$, $q_2 = 0·0$.

4. Iterative estimation procedure

The likelihood equations (2·3) and (2·4) have explicit solutions only in the case of complete samples ($m = n$). For censored samples, however, iterative procedures have been developed for finding the joint maximum-likelihood estimators. These involve estimating the parameters, one at a time, in the cyclic order μ, σ, omitting a parameter if it is assumed to be known. One starts by choosing initial estimate(s) for the unknown parameter(s). At each step, the rule of false position (iterative linear interpolation) is used to determine the value of the parameter then being estimated which satisfies the appropriate likelihood equation, in which the latest estimate (or known value) of the other parameter has been substituted. Iteration continues until the results of successive steps agree to within some assigned tolerance. Experience has shown that the rate of convergence is quite rapid if the initial estimates are reasonable and the amount of censoring is not excessive.

5. Monte Carlo study of maximum-likelihood estimators from small samples

There is no known analytic method of determining the variances and covariance of the joint distribution of the maximum-likelihood estimators $\hat{\mu}$ and $\hat{\sigma}$ from small samples. Furthermore, these estimators, while asymptotically unbiased, are known to be biased for small samples (except $\hat{\mu}$ when censoring is absent or symmetric), though analytic expressions for the bias are known only in the case of estimation from the complete sample ($m = n$). In order to obtain information about the small-sample properties of these estimators, a Monte Carlo study was performed on the IBM 7094 computer. For $n = 10$ and for $n = 20$, one thousand random samples of n standard normal deviates were generated, and the n deviates in each sample were arranged in order from smallest to largest. The iterative procedure described in §4 was used to compute the estimates $\hat{\mu}$ and $\hat{\sigma}$, also $\hat{\mu}|\sigma$ and $\hat{\sigma}|\mu$, from the m order statistics remaining in each sample after proportions q_1 and q_2 had been censored from below and from above, respectively, where q_1 and q_2 were taken at intervals of 0·1, subject to the restrictions $q_1 \leqslant q_2$ and $m \geqslant 2$. The means, variances, and covariances of the estimates from 1000 samples of size $n = 10$ are given in Table 2, and similar results for $n = 20$ are given in Table 3. There is no loss of generality associated with the restriction $q_1 \leqslant q_2$, since interchanging q_1 and q_2 would produce no change in the expected tabular values except that of reversing the signs of the mean of $\hat{\mu}$ and the covariance of $\hat{\mu}$ and $\hat{\sigma}$. The rows of Table 2 (and likewise Table 3) are not statistically independent, since they are based on the same samples (with different proportions censored).

Table 2. *Means, variances, and covariances of maximum-likelihood estimates of mean and standard deviation of standard normal population ($\mu = 0$, $\sigma = 1$) from 1000 samples of size $n = 10$ with proportions q_1 censored from below and q_2 from above*

q_1	q_2	$M(\hat\mu)$	$M(\hat\sigma)$	$V(\hat\mu)$	$C(\hat\mu, \hat\sigma)$	$V(\hat\sigma)$	$M(\hat\mu\|\sigma)$	$M(\hat\sigma\|\mu)$	$V(\hat\mu\|\sigma)$	$V(\hat\sigma\|\mu)$
0·0	0·0	−0·00	0·93	0·100	0·000	0·059	−0·00	0·98	0·100	0·048
·0	·1	−·01	·92	·101	·003	·057	−·00	·98	·101	·057
·0	·2	−·02	·90	·104	·009	·067	−·00	·97	·103	·067
·0	·3	−·04	·88	·114	·010	·078	−·01	·97	·106	·077
·0	·4	−·07	·85	·125	·028	·084	−·01	·96	·116	·081
·0	·5	−·11	·81	·146	·047	·101	−·02	·95	·124	·091
·0	·6	−·19	·76	·184	·075	·123	−·03	·94	·138	·101
·0	·7	−·31	·67	·265	·130	·166	−·03	·93	·166	·105
·0	·8	−·57	·51	·412	·210	·204	−·05	·93	·197	·108
0·1	0·1	−0·00	0·90	0·101	0·000	0·067	−0·00	0·97	0·101	0·067
·1	·2	−·01	·88	·105	·007	·078	−·00	·97	·104	·079
·1	·3	−·03	·86	·114	·018	·091	−·01	·96	·111	·093
·1	·4	−·07	·81	·126	·029	·101	−·01	·95	·117	·101
·1	·5	−·12	·75	·147	·052	·125	−·02	·93	·126	·117
·1	·6	−·22	·66	·187	·085	·152	−·03	·92	·140	·134
·1	·7	−·41	·49	·271	·143	·189	−·04	·90	·163	·139
0·2	0·2	−0·01	0·87	0·108	−0·001	0·096	−0·00	0·97	0·108	0·100
·2	·3	−·03	·83	·116	·011	·114	−·01	·96	·115	·122
·2	·4	−·06	·77	·127	·024	·129	−·02	·95	·122	·137
·2	·5	−·13	·65	·147	·051	·164	−·02	·92	·131	·169
·2	·6	−·27	·49	·181	·084	·188	−·03	·89	·146	·208
0·3	0·3	−0·01	0·79	0·119	0·003	0·145	−0·01	0·96	0·119	0·157
·3	·4	−·05	·69	·127	·020	·176	−·02	·94	·125	·196
·3	·5	−·13	·52	·148	·050	·217	−·02	·90	·136	·258
0·4	0·4	−0·01	0·61	0·134	0·000	0·218	−0·01	0·90	0·134	0·282

Table 3. *Means, variances, and covariances of maximum-likelihood estimates of mean and standard deviation of standard normal population ($\mu = 0$, $\sigma = 1$) from 1000 samples of size $n = 20$ with proportions q_1 censored from below and q_2 from above*

q_1	q_2	$M(\hat\mu)$	$M(\hat\sigma)$	$V(\hat\mu)$	$C(\hat\mu, \hat\sigma)$	$V(\hat\sigma)$	$M(\hat\mu\|\sigma)$	$M(\hat\sigma\|\mu)$	$V(\hat\mu\|\sigma)$	$V(\hat\sigma\|\mu)$
0·0	0·0	0·01	0·97	0·048	0·000	0·023	0·01	0·99	0·048	0·022
·0	·1	·01	·96	·048	·002	·027	·01	·99	·048	·027
·0	·2	−·00	·95	·050	·005	·032	·01	·99	·049	·031
·0	·3	−·01	·94	·054	·010	·038	·00	·98	·052	·036
·0	·4	−·03	·93	·061	·017	·045	−·00	·97	·055	·040
·0	·5	−·04	·91	·073	·029	·057	−·00	·97	·059	·048
·0	·6	−·07	·89	·095	·048	·075	−·00	·97	·064	·052
·0	·7	−·13	·86	·142	·083	·100	−·00	·96	·074	·055
·0	·8	−·27	·76	·257	·161	·141	−·02	·96	·095	·055
·0	·9	−·73	·52	·503	·289	·202	−·05	·97	·150	·059
0·1	0·1	0·01	0·96	0·049	−0·001	0·033	0·01	0·99	0·049	0·033
·1	·2	·00	·94	·051	·002	·039	·01	·98	·051	·038
·1	·3	−·01	·92	·054	·008	·048	·00	·98	·053	·045
·1	·4	−·03	·90	·061	·015	·057	·00	·97	·056	·052
·1	·5	−·05	·88	·074	·031	·076	·00	·99	·061	·063
·1	·6	−·09	·84	·097	·056	·104	·00	·95	·066	·073
·1	·7	−·18	·76	·152	·104	·146	−·00	·94	·077	·079
·1	·8	−·51	·49	·274	·183	·193	−·02	·93	·100	·080
0·2	0·2	0·01	0·93	0·052	−0·001	0·048	0·01	0·98	0·052	0·047
·2	·3	−·00	·91	·055	·006	·063	·00	·97	·054	·060
·2	·4	−·02	·87	·061	·014	·076	·00	·96	·058	·070
·2	·5	−·05	·83	·074	·034	·108	·00	·96	·063	·092
·2	·6	−·11	·75	·101	·072	·160	·00	·94	·069	·116
·2	·7	−·31	·51	·168	·143	·233	−·00	·91	·081	·133
0·3	0·3	0·01	0·88	0·057	−0·001	0·078	0·01	0·97	0·057	0·075
·3	·4	−·02	·83	·062	·009	·101	·00	·96	·061	·095
·3	·5	−·06	·74	·074	·032	·145	·00	·94	·067	·137
·3	·6	−·18	·50	·098	·073	·214	·00	·89	·073	·190
0·4	0·4	0·00	0·74	0·064	−0·000	0·152	0·00	0·94	0·064	0·148
·4	·5	−·06	·49	·074	·029	·220	·00	·90	·070	·258

$M(\hat\mu)$ = mean value of M.L.E. $\hat\mu$ (σ unknown);
$M(\hat\sigma)$ = mean value of M.L.E. $\hat\sigma$ (μ unknown);
$V(\hat\mu)$ = variance of M.L.E. $\hat\mu$ (σ unknown);
$C(\hat\mu, \hat\sigma)$ = covariance of M.L.E. $\hat\mu$ and $\hat\sigma$;
$V(\hat\sigma)$ = variance of M.L.E. $\hat\sigma$ (μ unknown);
$M(\hat\mu|\sigma)$ = mean value of M.L.E. $\hat\mu$ (σ known);
$M(\hat\sigma|\mu)$ = mean value of M.L.E. $\hat\sigma$ (μ known);
$V(\hat\mu|\sigma)$ = variance of M.L.E. $\hat\mu$ (σ known).
$V(\hat\sigma|\mu)$ = variance of M.L.E. $\hat\sigma$ (μ known).

Table 4. *Comparison of measures of precision of best linear unbi[illegible] estimators (blue) and maximum-likelihood estimators* (M.L.E.) *of parameters of normal population from samples of size $n = 10$ with proportions q_1 censored from below and q_2 from above*

q_1	q_2	$V(\mu^*)$	M.S.E. $(\hat{\mu})$	$AV(\hat{\mu})$	M.S.E. $(\hat{\mu}\mid\sigma)$	$AV(\hat{\mu}\mid\sigma)$	$V(\sigma^*)$	M.S.E. $(\hat{\sigma})$	$AV(\hat{\sigma})$	M.S.E. $(\hat{\sigma}\mid\mu)$	$AV(\hat{\sigma}\mid\mu)$
0·0	0·0	0·100	0·100	0·100	[illegible]	0·100	[illegible]	0·053	0·050	0·048	0·059
·0	·1	·102	·101	·102	·101	·102	·068	·064	·059	·057	·058
·0	·2	·107	·105	·100	·103	·105	·081	·076	·069	·068	·068
·0	·3	·117	·115	·114	·109	·109	·098	·092	·082	·078	·078
·0	·4	·134	·130	·127	·116	·114	·124	·104	·099	·082	·080
·0	·5	·166	·158	·152	·125	·122	·161	·136	·124	·094	·100
·0	·6	·237	·220	·190	·139	·134	·225	·184	·161	·105	·109
·0	·7	·417	·359	·302	·161	·153	·354	·269	·225	·110	·111
·0	·8	1·127	·733	·578	·166	·187	·746	·444	·354	·114	·115
0·1	0·1	0·104	0·101	0·103	0·101	0·103	0·082	0·078	0·070	0·068	0·070
·1	·2	·108	·105	·107	·104	·106	·101	·091	·085	·080	·084
·1	·3	·117	·115	·114	·111	·111	·128	·112	·104	·091	·101
·1	·4	·134	·130	·127	·117	·116	·168	·136	·132	·103	·126
·1	·5	·171	·162	·154	·126	·125	·237	·185	·174	·122	·140
·1	·6	·267	·237	·213	·141	·137	·377	·271	·246	·141	·159
·1	·7	·630	·440	·367	·164	·157	·807	·453	·305	·148	·169
0·2	0·2	0·111	0·108	0·110	0·108	0·110	0·120	0·114	0·105	0·101	0·105
·2	·3	·118	·117	·115	·115	·114	·171	·144	·134	·123	·133
·2	·4	·134	·131	·128	·122	·120	·243	·183	·178	·140	·168
·2	·5	·177	·164	·156	·132	·129	·388	·269	·254	·175	·210
·2	·6	·340	·256	·239	·147	·142	·832	·448	·409	·219	·253
0·3	0·3	0·122	0·119	0·119	0·119	0·119	0·214	0·194	0·180	0·156	0·186
·3	·4	·134	·130	·128	·125	·126	·392	·272	·257	·206	·251
·3	·5	·187	·165	·156	·136	·135	·843	·456	·416	·266	·350
0·4	0·4	0·138	0·134	0·133	0·134	0·133	0·846	0·457	0·417	0·292	0·417

Table 5. *Comparison of measures of precision of best linear unbiased estimators (blue) and maximum-likelihood estimators* (M.L.E.) *of parameters of normal population from samples of size $n = 20$ with proportions q_1 censored from below and q_2 from above*

q_1	q_2	$V(\mu^*)$	M.S.E. $(\hat{\mu})$	$AV(\hat{\mu})$	M.S.E. $(\hat{\mu}\mid\sigma)$	$AV(\hat{\mu}\mid\sigma)$	$V(\sigma^*)$	M.S.E. $(\hat{\sigma})$	$AV(\hat{\sigma})$	M.S.E. $(\hat{\sigma}\mid\mu)$	$AV(\hat{\sigma}\mid\mu)$
0·0	0·0	0·050	0·048	0·050	0·048	0·050	0·027	0·024	0·025	0·022	0·025
·0	·1	·051	·048	·051	·048	·051	·032	·029	·029	·027	·029
·0	·2	·053	·050	·053	·049	·052	·037	·034	·034	·031	·034
·0	·3	·058	·054	·057	·052	·054	·045	·042	·041	·039	·039
·0	·4	·065	·061	·064	·055	·057	·055	·050	·050	·041	·045
·0	·5	·079	·075	·076	·059	·061	·070	·065	·062	·047	·050
·0	·6	·108	·100	·099	·064	·067	·094	·087	·081	·053	·054
·0	·7	·178	·158	·151	·074	·076	·139	·122	·122	·056	·057
·0	·8	·383	·331	·289	·095	·094	·244	·198	·177	·057	·057
·0	·9	1·871	1·089	·080	·152	·139	·786	·439	·379	·060	·059
0·1	0·1	0·052	0·049	0·052	0·049	0·052	0·038	0·035	0·035	0·033	0·035
·1	·2	·054	·051	·053	·051	·053	·046	·042	·042	·038	·042
·1	·3	·058	·054	·057	·053	·055	·057	·054	·052	·046	·050
·1	·4	·065	·061	·064	·056	·058	·074	·067	·066	·053	·060
·1	·5	·081	·076	·077	·061	·062	·100	·091	·087	·064	·070
·1	·6	·118	·105	·106	·066	·068	·140	·130	·123	·075	·079
·1	·7	·228	·185	·183	·077	·078	·266	·205	·198	·082	·084
·1	·8	·934	·531	·480	·106	·097	·873	·456	·433	·084	·086
0·2	0·2	0·055	0·052	0·055	0·052	0·055	0·058	0·053	0·053	0·047	0·053
·2	·3	·058	·055	·058	·055	·057	·075	·071	·067	·061	·066
·2	·4	·065	·061	·064	·058	·060	·103	·092	·089	·072	·084
·2	·5	·082	·077	·078	·063	·065	·154	·130	·127	·094	·105
·2	·6	·134	·114	·115	·069	·071	·275	·220	·205	·120	·126
·2	·7	·456	·265	·259	·081	·082	·898	·470	·446	·142	·141
0·3	0·3	0·060	0·058	0·059	0·058	0·059	0·104	0·092	0·090	0·076	0·090
·3	·4	·066	·062	·064	·061	·063	·156	·130	·128	·097	·120
·3	·5	·084	·077	·078	·067	·068	·270	·211	·208	·140	·180
·3	·6	·201	·131	·134	·073	·075	·908	·462	·452	·202	·252
0·4	0·4	0·068	0·064	0·067	0·064	0·067	0·280	0·220	0·209	0·152	0·209
·4	·5	·087	·077	·078	·070	·072	·913	·478	·454	·268	·417

$V(\mu^*)$ = Exact variance of blue μ^*;
M.S.E.$(\hat{\mu})$ = Mean square error of M.L.E. $\hat{\mu}$ in 1000 samples (σ unknown);
$AV(\hat{\mu})$ = Variance of M.L.E. $\hat{\mu}$ as given by asymptotic formula (σ unknown).
M.S.E.$(\hat{\mu}\mid\sigma)$ = Mean square error of M.L.E. $\hat{\mu}$ in 1000 samples (σ known);
$AV(\hat{\mu}\mid\sigma)$ = Variance of M.L.E. $\hat{\mu}$ as given by asymptotic formula (σ known);
$V(\sigma^*)$ = Exact variance of blue σ^*;
M.S.E.$(\hat{\sigma})$ = Mean square error of M.L.E. $\hat{\sigma}$ in 1000 samples (μ unknown);
$AV(\hat{\sigma})$ = Variance of M.L.E. $\hat{\sigma}$ as given by asymptotic formula (μ unknown).
M.S.E.$(\hat{\sigma}\mid\mu)$ = Mean square error of M.L.E. $\hat{\sigma}$ in 1000 samples (μ known);
$AV(\hat{\sigma}\mid\mu)$ = Variance of M.L.E. $\hat{\sigma}$ as given by asymptotic formula (μ known)

The following tentative conclusions may be drawn from Tables 2 and 3: (1) When $q_1 < q_2$, the estimates $\hat{\mu}$ and $\hat{\mu}|\sigma$ are negatively biased. (By symmetry, these estimates are positively biased when $q_1 > q_2$ and unbiased when $q_1 = q_2$.) (2) The estimates $\hat{\sigma}$ and $\hat{\sigma}|\mu$ are negatively biased regardless of the relative magnitude of q_1 and q_2. (3) The bias in estimating either parameter is much smaller when the other parameter is known than it is when both parameters are being estimated simultaneously. (4) The bias of $\hat{\sigma}$ (μ unknown) is approximately equal to $-1/m$.

It would be desirable to compare the variances of μ and σ from samples of sizes 10 and 20 with the values which one would obtain by substituting $n = 10$ and $n = 20$ in the asymptotic values given in Table 1, as well as with the variances of the best linear unbiased estimators μ^* and σ^*. Direct comparison of variances of estimators is appropriate, however, only when all the estimators are unbiased. In order to compensate for the bias in the maximum-likelihood estimators, the mean square errors of $\hat{\mu}$, $\hat{\sigma}$, $\hat{\mu}|\sigma$, and $\hat{\sigma}|\mu$ were computed. These were compared with the variances of the best linear unbiased estimators given by Sarhan & Greenberg (1962, Table 10C 2) and with the variances of the maximum-likelihood estimators given by the asymptotic formula, which were obtained by dividing by n the values given in Table 1. The results are shown in Tables 4 and 5 from which the following tentative conclusions may be drawn: (1) The precision of the maximum-likelihood estimator $\hat{\mu}$, when proper allowance is made for bias, closely approximates that predicted by the aysmptotic formula for the variance of $\hat{\mu}$, even for m as small as 2, except in cases of strongly asymmetric censoring. (2) The precision of the maximum-likelihood estimator $\hat{\sigma}$, when proper allowance is made for bias, closely approximates that predicted by the asymptotic formula for the variance of $\hat{\sigma}$, except when m is quite small and/or censoring is strongly asymmetric. (3) Maximum-likelihood estimators tend to be somewhat more precise than best linear unbiased estimators. The difference is greatest for estimates of μ in cases of strongly asymmetric censoring and for estimates of σ when m is small and/or censoring is strongly asymmetric.

Approximate corrections for the bias of the maximum-likelihood estimators $\hat{\mu}$, $\hat{\sigma}$, $\hat{\mu}|\sigma$, and $\hat{\sigma}|\sigma$ for $n = 10$ and $n = 20$ can be made by use of the means found in the Monte Carlo study and recorded in Tables 2 and 3.

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Iterative procedures are given for joint maximum-likelihood estimation based on singly and doubly censored samples from a normal population. The simultaneous equations yielding the maximum-likelihood estimates are obtained. Since their algebraic solution is impossible, iterative procedures are proposed which are applicable in the most general case in which both parameters are unknown and in special cases in which either of the parameters is known. The asymptotic variances and covariances are tabulated for 10% censoring intervals. A Monte Carlo investigation of the means and standard diviations of the maximum-likelihood estimators was made for 1000 samples from the standard normal population for $n = 10$ and $n = 20$. A comparison was then made of best linear unbiased estimators and maximum-likelihood estimators for $n = 10$ and $n = 20$.

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